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Characteristic Length and Dynamic Time Scale Associated with Aircraft Pitching Motion

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The 5%-static-margin requirement once used by the military has been removed in favor of dynamic requirements. However, static margin is still commonly used as a preliminary guideline for establishing adequate pitch stability in piloted airplanes. Here, a dynamic margin is defined to be the distance that the airplane's stick-fixed maneuver point lies aft of the center of gravity, expressed as a fraction of the pitch radius of gyration. A proposed minimum-dynamic-margin constraint is shown to agree with the short-period-frequency requirements used in recent military specifications to an accuracy of about 2%. For some airplanes, this stability constraint is shown to differ from the 5%-minimum-static-margin guideline by more than 400%. Two worked examples are presented, which demonstrate the significance of the proposed dynamic-margin constraint.

Nomenclature

a = axial distance aft from some arbitrary reference point
b = wingspan
$C_{L,\alpha}$ = change in lift coefficient with angle of attack
$C_{m_{\mathrm{ac}}}$ = wing moment coefficient based on arbitrary reference chord length
$\tilde{C}_{m_{\mathrm{ac}}}$ = airfoil section moment coefficient based on local chord length
$C_{m,\bar{q}}$ = change in traditional moment coefficient with traditional dimensionless pitch rate
$C_{m,\alpha}$ = change in traditional moment coefficient with angle of attack
$C_{\breve{m}}$ = dynamic moment coefficient based on pitch radius of gyration
$C_{\breve{m},\breve{q}}$ = change in dynamic moment coefficient with dynamic pitch rate
$C_{\breve{m},\alpha}$ = change in dynamic moment coefficient with angle of attack
C_W = weight coefficient
c = local airfoil section chord length
$\bar{c}$ = geometric mean chord length, S/b
c_f = fuselage chord length
$\bar{c}_{\mathrm{mac}}$ = mean aerodynamic chord length, defined by Eq. (16)
c_{ref} = arbitrary reference chord length
D = drag force
g = acceleration of gravity
h_{np} = normal distance upward from the center of gravity to the neutral point
h_T = normal distance upward from the center of gravity to the center of thrust
I_{yy} = aircraft pitching moment of inertia about the center of gravity
L = lift force
L_0 = lift force with $q=\alpha=\dot{\alpha}=\delta_e=0$
$L_{,q}$ = change in lift force with respect to pitch rate
$L_{,\breve{q}}$ = change in lift force with respect to dynamic pitch rate
$L_{,\alpha}$ = change in lift force with respect to angle of attack
$L_{,\dot{\alpha}}$ = change in lift force with respect to angle-of-attack rate
$L_{,\breve{\dot{\alpha}}}$ = change in lift force with respect to dynamic angle-of-attack rate
$L_{,\delta_e}$ = change in lift force with respect to elevator deflection
l_{np} = axial distance aft from the center of gravity to the stick-fixed neutral point
l_{mp} = axial distance aft from the center of gravity to the stick-fixed maneuver point
M = aircraft mass
m = pitching moment about the center of gravity, positive nose up
m_{np} = pitching moment about the neutral point or aerodynamic center, positive nose up
$m_{\mathrm{np}0}$ = pitching moment about the neutral point with $q=\dot{\alpha}=\delta_e=0$
$m_{\mathrm{np},q}$ = change in pitching moment about the neutral point with respect to pitch rate
$m_{\mathrm{np},\dot{\alpha}}$ = change in pitching moment about the neutral point with respect to angle-of-attack rate
m_{np,δ_e} = change in pitching moment about the neutral point with respect to elevator deflection
m_0 = pitching moment about the center of gravity with $q=\alpha=\dot{\alpha}=\delta_e=0$
$m_{,q}$ = change in pitching moment about the center of gravity with respect to pitch rate
$m_{,\breve{q}}$ = change in pitching moment about the center of gravity with respect to dynamic pitch rate
$m_{,\alpha}$ = change in pitching moment about the center of gravity with respect to angle of attack
$m_{,\dot{\alpha}}$ = change in pitching moment about the center of gravity with respect to angle-of-attack rate
$m_{,\delta_e}$ = change in pitching moment about the center of gravity with respect to elevator deflection
n = load factor, L/W
q = pitch rate, positive nose rising
$\dot{q}$ = change in pitch rate with respect to time

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$\bar{q}$ = traditional dimensionless pitch rate, $c_{\text{ref}}q/(2V_o)$
$\breve{q}$ = dimensionless dynamic pitch rate, $V_o q/g$
$\dot{q}_c$ = experimental constraint based on pilot pitch-acceleration-sensitivity limits and flight-phase requirements
R_A = wing aspect ratio, b^2/S
r_{xx}, r_{yy}, r_{zz} = aircraft roll, pitch, and yaw radii of gyration about the center of gravity
S = wing planform area
T = thrust force
u = forward component of the aircraft velocity parallel with the fuselage reference line
$\dot{u}$ = change in forward component of the aircraft velocity with respect to time
V = magnitude of aircraft velocity
$\dot{V}$ = change in magnitude of aircraft velocity with respect to time
V_o = magnitude of equilibrium aircraft velocity
W = aircraft weight
w = downward component of the aircraft velocity normal to the fuselage reference line
$\dot{w}$ = change in downward component of the aircraft velocity with respect to time
X = forward component of the aerodynamic force parallel with the fuselage reference line
x, y, z = axial, spanwise, and normal coordinates relative to the center of gravity
Z = downward component of the aerodynamic force normal to the fuselage reference line
α = freestream angle of attack relative to the fuselage reference line, positive nose up
$\dot{\alpha}$ = change in freestream angle of attack with respect to time
$\breve{\dot{\alpha}}$ = dimensionless dynamic angle-of-attack rate, $V_o\dot{\alpha}/g$
δ_e = elevator deflection angle
ζ = short-period damping ratio
θ = elevation angle between the horizontal and the fuselage reference line, positive nose up
λ, λ_1, λ_2 = short-period eigenvalues
ρ = freestream air density
$\bar{\tau}$ = traditional reference time scale, $c_{\text{ref}}/(2V_o)$
$\breve{\tau}$ = characteristic dynamic time scale, V_o/g
ω_n = short-period undamped natural frequency

I. Introduction

IN 1911, less than a decade after the Wright brothers' first successful powered flight, the British mathematician George H. Bryan [1] was the first to present the aircraft equations of motion in the form that is familiar today. Although others contributed, it was Bryan who provided the mathematical framework that was needed for the analysis of flight dynamics. Bryan introduced the concept of *stability derivatives* and he divided an airplane's motion into what are now referred to as the longitudinal and lateral components. Furthermore, Bryan exposed the nature of an airplane's natural frequencies. Although Bryan introduced the concept of static stability, he also recognized that static stability was not an absolute requirement for human-piloted airplanes. A review of the work by Bryan and others in the early history of the development of airplane stability and control theory was presented by the famous aeronautical engineer, Courtland D. Perkins, in his 1970 von Kármán lecture [2].

One traditional measure of an airplane's static stability in pitch is the static margin, which is defined to be the distance that the airplane's neutral point lies aft of the center of gravity, expressed as a fraction of the reference chord length for the wing. As pointed out by Perkins [2], many of the early successful airplanes, including the 1903 Wright Flyer, had slightly negative static margins. Perkins stated that the Wright brothers' success, with an unstable airplane, was due to the fact that they "recognized that powerful controls would permit them to maintain the desired equilibrium." In spite of early successes with statically unstable airplanes and Bryan's contention that static stability was not an absolute requirement [1], a positive static margin was long considered by most designers to be a requirement for human-piloted airplanes. For a more detailed discussion of static margin, see Etkin and Reid [3], Nelson [4], Pamadi [5], Phillips [6], or Yechout [7].

Before MIL-8785 [8], U.S. military specifications required a minimum static margin of 5%. However, this requirement was removed by MIL-8785, because research had shown that aircraft handling qualities do not scale with static margin (see MIL-1797 [9] for further details). In both MIL-8785 and MIL-1797, the 5%-static-margin requirement was replaced with dynamic stability requirements developed from aircraft handling-qualities theory and experimentation [8–11]. In spite of the fact that the 5%-static-margin requirement was removed by the military, it is still commonly used in the private sector as a preliminary guideline for establishing adequate handling qualities for piloted airplanes.

Longitudinal aircraft handling qualities depend significantly on the aircraft's dynamic response in pitch to control inputs and atmospheric disturbances. Significant measures of an aircraft's dynamic pitch response include the elevator angle per g, as well as the short-period natural frequency and damping ratio. The elevator angle per g is defined to be the change in elevator deflection with respect to load factor for a constant-speed pull-up maneuver. The short-period mode is a heavily damped oscillatory mode that involves rapid changes in angle of attack and vertical displacement at nearly constant airspeed. Because both the elevator angle per g and the short-period mode are associated with a dynamic response at constant or nearly constant airspeed, these two measures of an airplane's pitch response are governed by the same two coupled differential equations. These are 1) the normal component of Newton's second law in the longitudinal plane, and 2) the longitudinal component of the angular momentum equation. For further details, see Etkin and Reid [3], Nelson [4], Pamadi [5], Phillips [6], or Yechout [7].

II. Linearized Pitch Dynamics

The distribution of longitudinal aerodynamic loads acting on an airplane can be replaced with a normal force, an axial force, and a pitching moment acting at the neutral point or aerodynamic center of the airplane, as shown in Fig. 1. Because the orientation of the fuselage reference line is arbitrary, here it is defined to pass through the center of gravity parallel with the thrust vector. Thus, the longitudinal components of Newton's second law and the longitudinal angular momentum equation about the center of gravity yield [3–7]

$$\begin{Bmatrix} M(\dot{u}+wq) \\ M(\dot{w}-uq) \\ I_{yy}\dot{q} \end{Bmatrix} = \begin{Bmatrix} X+T-W\sin\theta \\ Z+W\cos\theta \\ m_{\text{np}}+l_{\text{np}}Z-h_{\text{np}}X-h_T T \end{Bmatrix} \tag{1}$$

where $\dot{u}$, $\dot{w}$, and $\dot{q}$ denote time derivatives.

An accurate approximation for aircraft pitch dynamics is obtained by neglecting changes in forward velocity and dropping the axial momentum equation. Thus, after expressing the normal and axial forces in terms of lift and drag ($Z=-L\cos\alpha-D\sin\alpha$ and $X=-D\cos\alpha+L\sin\alpha$), Eq. (1) can be approximated with the second-order system

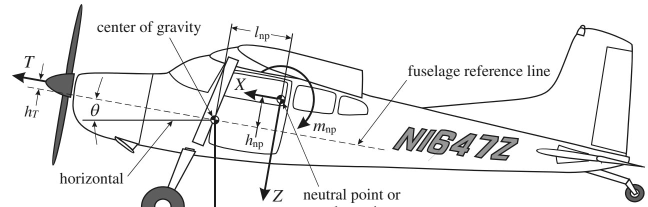


Fig. 1 Resultant longitudinal aerodynamic force and moment components acting at the neutral point.

$$\begin{Bmatrix} M\dot{w} \\ I_{yy}\dot{q} \end{Bmatrix} = \begin{Bmatrix} W\cos\theta + Muq \\ m_{\text{np}} - h_T T \end{Bmatrix} - \begin{bmatrix} \cos\alpha & \sin\alpha \\ (l_{\text{np}}\cos\alpha + h_{\text{np}}\sin\alpha) & (l_{\text{np}}\sin\alpha - h_{\text{np}}\cos\alpha) \end{bmatrix} \begin{Bmatrix} L \\ D \end{Bmatrix} \quad (2)$$

For small deviations from level flight, the angle of attack α and the elevation angle θ are small. Furthermore, drag contributes little to the normal force and pitching moment, and for most applications, the vertical offsets are small. Thus, after applying the small-angle approximations ($\cos\theta \cong 1$, $\cos\alpha \cong 1$, $u = V\cos\alpha \cong V$, and $w = V\sin\alpha \cong V\alpha$), we neglect drag and vertical offsets so that Eq. (2) is closely approximated as

$$\begin{Bmatrix} M(V\dot{\alpha} + \alpha\dot{V}) \\ I_{yy}\dot{q} \end{Bmatrix} = \begin{Bmatrix} W - L + MVq \\ m_{\text{np}} - l_{\text{np}}L \end{Bmatrix} \quad (3)$$

For small values of α, q, and δ_e, the lift is closely approximated as a linear function of α, $\dot{\alpha}$, q, and δ_e. By definition, the pitching moment about the neutral point does not vary with small changes in α. Thus, m_{np} is closely approximated as a linear function of only $\dot{\alpha}$, q, and δ_e Accordingly, we use

$$\begin{Bmatrix} L \\ m_{\text{np}} \end{Bmatrix} = \begin{Bmatrix} L_0 + L_{,\alpha}\alpha + L_{,\dot{\alpha}}\dot{\alpha} + L_{,q}q + L_{,\delta_e}\delta_e \\ m_{\text{np}0} + m_{\text{np},\dot{\alpha}}\dot{\alpha} + m_{\text{np},q}q + m_{\text{np},\delta_e}\delta_e \end{Bmatrix}$$

and with the airspeed held constant at V_o, Eq. (3) becomes

$$\begin{bmatrix} (MV_o + L_{,\dot{\alpha}}) & 0 \\ (l_{\text{np}}L_{,\dot{\alpha}} - m_{\text{np},\dot{\alpha}}) & I_{yy} \end{bmatrix} \begin{Bmatrix} \dot{\alpha} \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} L_{,\alpha} & (L_{,q} - MV_o) \\ l_{\text{np}}L_{,\alpha} & (l_{\text{np}}L_{,q} - m_{\text{np},q}) \end{bmatrix} \begin{Bmatrix} \alpha \\ q \end{Bmatrix} = \begin{Bmatrix} W - L_0 \\ m_{\text{np}0} - l_{\text{np}}L_0 \end{Bmatrix} + \begin{Bmatrix} -L_{,\delta_e} \\ m_{\text{np},\delta_e} - l_{\text{np}}L_{,\delta_e} \end{Bmatrix}\delta_e$$

After using the definitions $m_{,\dot{\alpha}} \equiv m_{\text{np},\dot{\alpha}} - l_{\text{np}}L_{,\dot{\alpha}}$, $m_{,q} \equiv m_{\text{np},q} - l_{\text{np}}L_{,q}$, $m_0 \equiv m_{\text{np}0} - l_{\text{np}}L_0$, and $m_{,\delta_e} \equiv m_{\text{np},\delta_e} - l_{\text{np}}L_{,\delta_e}$, this yields a small-angle approximation for the longitudinal equations of motion at constant airspeed and small deviations from steady-level flight,

$$\begin{bmatrix} (MV_o + L_{,\dot{\alpha}}) & 0 \\ -m_{,\dot{\alpha}} & I_{yy} \end{bmatrix} \begin{Bmatrix} \dot{\alpha} \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} L_{,\alpha} & (L_{,q} - MV_o) \\ l_{\text{np}}L_{,\alpha} & -m_{,q} \end{bmatrix} \begin{Bmatrix} \alpha \\ q \end{Bmatrix} = \begin{Bmatrix} W - L_0 \\ m_0 \end{Bmatrix} + \begin{Bmatrix} -L_{,\delta_e} \\ m_{,\delta_e} \end{Bmatrix}\delta_e \quad (4)$$

It should be noted that the derivatives $L_{,\dot{\alpha}}$ and $L_{,q}$ are not typically included in the linearized formulation used to develop the traditional short-period approximation [3–7] because they contribute little to the pitch dynamics of conventional piloted airplanes. These terms have been included in Eq. (4) so that the magnitude of their contribution may be quantitatively assessed. The traditional measures of an aircraft's dynamic pitch response are easily estimated directly from Eq. (4).

A. Elevator Angle per *g*

The elevator angle per g is defined to be the change in elevator deflection with respect to load factor for the constant-speed pull-up maneuver, which is shown in Fig. 2. By definition, this is a constant-speed maneuver, and so both of the body-fixed components of airspeed and the angular rate are held constant, $\dot{\alpha} = \dot{q} = 0$. Hence, the constant-speed pull-up maneuver is the steady particular solution to Eq. (4), which is easily rearranged to give

$$\begin{bmatrix} L_{,\alpha} & L_{,\delta_e} \\ l_{\text{np}}L_{,\alpha} & -m_{,\delta_e} \end{bmatrix} \begin{Bmatrix} \alpha \\ \delta_e \end{Bmatrix} = \begin{Bmatrix} W - L_0 + (MV_o - L_{,q})q \\ m_0 + m_{,q}q \end{Bmatrix} \quad (5)$$

Equation (5) is readily solved for the angle of attack and elevator deflection required to support a given pitch rate in a constant-speed pull-up maneuver:

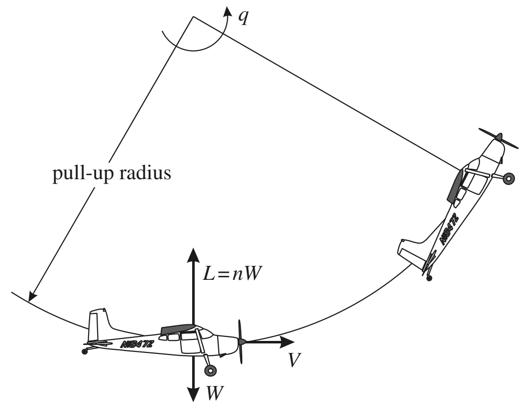


Fig. 2 Constant-speed pull-up maneuver.

$$\alpha = \frac{[W - L_0 + (MV_o - L_{,q})q]m_{,\delta_e} + (m_0 + m_{,q}q)L_{,\delta_e}}{L_{,\alpha}(m_{,\delta_e} + L_{,\delta_e}l_{\text{np}})} \quad (6)$$

$$\delta_e = \frac{[W - L_0 + (MV_o - L_{,q})q]l_{\text{np}} - m_0 - m_{,q}q}{m_{,\delta_e} + L_{,\delta_e}l_{\text{np}}} \quad (7)$$

The centripetal acceleration can be written in terms of load factor, that is, $V_o q = (n-1)g$. Thus, the pitch rate in Eqs. (6) and (7) can also be expressed in terms of the load factor:

$$q = (n-1)g/V_o \quad (8)$$

The elevator angle per g is found by differentiating Eq. (7) with respect to the load factor. Using the definition $W \equiv Mg$ together with Eq. (8), this gives

$$\frac{\partial\delta_e}{\partial n} = \frac{(MV_o - L_{,q})l_{\text{np}} - m_{,q}}{m_{,\delta_e} + L_{,\delta_e}l_{\text{np}}}\frac{\partial q}{\partial n} = \frac{(W - L_{,q}g/V_o)l_{\text{np}} - m_{,q}g/V_o}{m_{,\delta_e} + L_{,\delta_e}l_{\text{np}}} \quad (9)$$

The significance of this result is discussed in typical textbooks on flight mechanics [3–7].

B. Short-Period Approximation

An accurate short-period approximation can be obtained from the eigenvalues for the homogeneous solution to Eq. (4), which results in the quadratic characteristic equation

$$\left| \begin{bmatrix} (MV_o + L_{,\dot{\alpha}}) & 0 \\ -m_{,\dot{\alpha}} & I_{yy} \end{bmatrix}\lambda + \begin{bmatrix} L_{,\alpha} & (L_{,q} - MV_o) \\ l_{\text{np}}L_{,\alpha} & -m_{,q} \end{bmatrix} \right| = (MV_o + L_{,\dot{\alpha}})I_{yy}\lambda^2 + [I_{yy}L_{,\alpha} - (MV_o + L_{,\dot{\alpha}})m_{,q} - (MV_o - L_{,q})m_{,\dot{\alpha}}]\lambda + [l_{\text{np}}(MV_o - L_{,q}) - m_{,q}]L_{,\alpha} = 0$$

and yields the short-period eigenvalues

$$\lambda = \frac{m_{,q}}{2I_{yy}} - \frac{L_{,\alpha} - (MV_o - L_{,q})m_{,\dot{\alpha}}/I_{yy}}{2(MV_o + L_{,\dot{\alpha}})} \pm \left\{ \left[\frac{m_{,q}}{2I_{yy}} - \frac{L_{,\alpha} - (MV_o - L_{,q})m_{,\dot{\alpha}}/I_{yy}}{2(MV_o + L_{,\dot{\alpha}})} \right]^2 - \frac{[l_{\text{np}}(MV_o - L_{,q}) - m_{,q}]L_{,\alpha}}{I_{yy}(MV_o + L_{,\dot{\alpha}})} \right\}^{1/2} \quad (10)$$

This gives the short-period undamped natural frequency

$$\omega_n = \sqrt{\lambda_1\lambda_2} = \sqrt{\frac{[l_{\text{np}}(MV_o - L_{,q}) - m_{,q}]L_{,\alpha}}{I_{yy}(MV_o + L_{,\dot{\alpha}})}} \quad (11)$$

and the short-period damping ratio

$$\zeta = -\frac{\lambda_1 + \lambda_2}{2\omega_n}) = \frac{1}{2\omega_n}\left[\frac{L_{,\alpha} - (MV_o - L_{,q})m_{,\dot{\alpha}}/I_{yy}}{(MV_o + L_{,\dot{\alpha}})} - \frac{m_{,q}}{I_{yy}}\right] \quad (12)$$

Applying the additional approximations $L_{,\dot{\alpha}} \cong L_{,q} \cong 0$ to Eqs. (11) and (12) produces a result consistent with the traditional short-period approximation [3–7]. For conventional piloted airplanes, this approximation agrees closely with more exact numerical solutions obtained from the full linearized longitudinal equations of motion.

III. Traditional Reference Length and Time Scale

As shown in Eq. (4), the pitching motion of an airplane is affected significantly by the forces of lift and weight, as well as the pitching moment. The lift and weight are traditionally nondimensionalized with respect to the product of the freestream dynamic pressure and the wing planform area to produce the conventional lift and weight coefficients. To nondimensionalize the pitching moment, an additional length scale is needed. In the early days of aeronautics, wings were typically rectangular and the wing chord length was chosen as the standard reference length used to define the pitching moment coefficient. If the wing chord length is not constant over the wingspan, some type of mean chord length is traditionally used as the reference length.

A. Wing Aspect Ratio and Geometric Mean Chord Length

Because wing aspect ratio has a profound effect on wing performance, both the wingspan and the geometric mean chord length

$$\bar{c} \equiv \frac{b}{R_A} \equiv \frac{S}{b} = \frac{1}{b}\int_{y=-b/2}^{b/2} c\,\mathrm{d}y \quad (13)$$

are important characteristic length scales associated with the static performance of a wing. This geometric mean chord length has long been used as a reference length for defining the wing pitching moment coefficient (for example, see Anderson [12,13]). Nevertheless, we see from Eq. (4) that the wing chord length does not appear directly in the equations of motion that govern the pitch dynamics of an airplane. Thus, we should not expect the effects of pitch dynamics on aircraft handling qualities to scale with the geometric mean chord length defined by Eq. (13).

B. Wing Camber and Mean Aerodynamic Chord Length

The pitching moment coefficient for a wing about its aerodynamic center depends on the spanwise distribution of wing camber:

$$(C_{m_{\mathrm{ac}}})_{\mathrm{wing}} = \frac{1}{Sc_{\mathrm{ref}}}\int_{y=-b/2}^{b/2} \tilde{C}_{m_{\mathrm{ac}}} c^2\,\mathrm{d}y \quad (14)$$

For wings with no control surface deflection or any other form of aerodynamic twist, the section moment coefficient is constant over the wingspan and Eq. (14) becomes

$$(C_{m_{\mathrm{ac}}})_{\mathrm{wing}} = \frac{\tilde{C}_{m_{\mathrm{ac}}}}{Sc_{\mathrm{ref}}}\int_{y=-b/2}^{b/2} c^2\,\mathrm{d}y \quad (15)$$

Equation (15) leads to the definition of a reference length scale associated with wing camber, which has come to be known as the mean aerodynamic chord length (e.g., Etkin and Reid [14] or Raymer [15])

$$\bar{c}_{\mathrm{mac}} \equiv \frac{1}{S}\int_{y=-b/2}^{b/2} c^2\,\mathrm{d}y \quad (16)$$

Early in the history of aeronautics, the mean aerodynamic chord length was defined differently. In a 1933 report on aeronautics nomenclature [16], the mean aerodynamic chord is defined as "The chord of an imaginary airfoil which would have force vectors throughout the flight range identical with those of the actual wing or wings." This is clearly an aerodynamic mean that depends on the lift distribution as well as the wing planform. It is not the camber-related mean chord length defined by Eq. (16). Following a great deal of wind-tunnel testing by NACA, in 1942, Diehl [17] quotes exactly the same definition for the mean aerodynamic chord as that given in [16] and he states, "No mean chord has been found that is better than a simple average mean chord so located at the centroid of the semispan as to have its quarter-chord point coinciding with the geometrical average quarter-chord point for the semispan." Based on these results, Diehl [17] recommended using the mean chord defined by Eq. (13) as a *best approximation* for the mean aerodynamic chord defined in [16]. In spite of this fact, a large fraction of the modern aeronautics community uses the camber-related mean chord length defined by Eq. (16) as a reference length for defining the pitching moment coefficient for a wing or airplane. Moreover, this mean aerodynamic chord appears as the reference length and location for the operational weight and balance calculations for many civil and military airplanes. The authors are not sure when the chord length defined by Eq. (16) was first used to define a pitching moment coefficient. In the late 1940s, the mean chord length defined by Eq. (13) was still being used consistently by NACA [18,19]. However, by the early 1950s, the definition given by Eq. (16) was beginning to appear in the NACA literature [20,21]. Because both pitch stability and pitch damping are independent of wing camber, we should not expect the effects of pitch dynamics on aircraft handling qualities to scale with the camber-related mean chord length that is defined by Eq. (16).

C. Traditional Reference Time Scale

The pitching moment about the aerodynamic center of a wing or complete airplane depends on the pitching rate, which has the units of inverse time. The reference time scale that is traditionally used to nondimensionalize the pitching rate is one-half the reference length scale divided by the airspeed:

$$\bar{\tau} \equiv c_{\mathrm{ref}}/(2V_o) \quad (17)$$

Because tradition is the only reason commonly given for using this reference time scale, its physical significance and exact origin are not known to the authors. However, this reference time scale was consistently used by NACA to nondimensionalize the pitching rate of an airplane for the analysis or testing of longitudinal dynamics throughout the 1940s and 1950s (e.g., [18–21]). Furthermore, it has since been used almost exclusively for this purpose up to the present time (e.g., [3–7]).

Returning again to Eq. (4), we emphasize that the wing chord length does not appear directly in the equations of motion that govern the pitch dynamics of an airplane. Thus, we should not use wing chord to define either the characteristic length scale or the characteristic time scale, which are used to nondimensionalize the pitching moment and pitching rate, if we expect the aircraft handling qualities to scale with the resulting dimensionless parameters. In the following sections, we shall examine the common measures of pitch dynamics in an attempt to identify more appropriate length and time scales that are associated with the pitch dynamics of an airplane.

IV. Maneuver Point, Dynamic Time Scale, and Dynamic Pitch Rate

The stick-fixed maneuver point for an airplane is defined as the center-of-gravity location that would force the elevator angle per g to zero. By definition, l is used here to represent an axial distance measured aft of the center of gravity (c.g.) and a is used to denote an axial distance measured aft of an arbitrary reference point. Thus, we can write $l_{\mathrm{np}} \equiv a_{\mathrm{np}} - a_{\mathrm{c.g.}}$ and Eq. (9) can be written

$$\frac{\partial \delta_e}{\partial n} = \frac{(W - L_{,q}g/V_o)(a_{\mathrm{np}} - a_{\mathrm{c.g.}}) - m_{,q}g/V_o}{m_{,\delta_e} + L_{,\delta_e}(a_{\mathrm{np}} - a_{\mathrm{c.g.}})}$$

With the center of gravity located at the stick-fixed maneuver point and the elevator angle per g set to zero, we have

$$(W - L_{,q}g/V_o)(a_{\mathrm{np}} - a_{\mathrm{mp}}) - m_{,q}g/V_o = 0$$

or $a_{\mathrm{mp}} = a_{\mathrm{np}} - (m_{,q}g/V_o)/(W - L_{,q}g/V_o)$. Thus, after subtracting $a_{\mathrm{c.g.}}$ from both sides of this relation, the distance aft from the actual center of gravity to the stick-fixed maneuver point is given by

$$l_{\mathrm{mp}} = l_{\mathrm{np}} - \frac{1}{V_o/g}\frac{\partial m}{\partial q} \bigg/ \left(W - \frac{1}{V_o/g}\frac{\partial L}{\partial q}\right) \tag{18}$$

Because l_{mp}, l_{np}, m/W, and m/L all have the units of length, Eq. (18) provides no clue to the dynamic length scale associated with pitch dynamics. However, a dynamic time scale associated with aircraft acceleration does present itself in Eq. (18), that is,

$$\text{dynamic time scale} \equiv \breve{\tau} \equiv V_o/g \tag{19}$$

Using this dynamic time scale, we can now define a dimensionless dynamic pitch rate as

$$\text{dynamic pitch rate} \equiv \breve{q} \equiv V_o q/g \tag{20}$$

Notice that this dimensionless pitch rate is the ratio of the centripetal acceleration to the gravitational acceleration, which is clearly an important dimensionless parameter associated with aircraft pitch dynamics. The dynamic pitch rate defined by Eq. (20) is proportional to airspeed, whereas the traditional dimensionless pitch rate varies inversely with airspeed. The dimensionless pitch rate defined by Eq. (20) is more characteristic of aircraft dynamics because the inertial effects of a given pitch rate increase with airspeed. Another important feature that should be noted about the time scale defined by Eq. (19) is that this time scale is completely independent of any length scale that may be characteristic of some other aspect of aircraft flight dynamics. Thus, the use of this dynamic time scale would not necessarily be confined to longitudinal dynamics. Using the definition from Eq. (20) in Eq. (18), the stick-fixed maneuver point for an airplane can be equivalently located from the relation

$$l_{\mathrm{mp}} = l_{\mathrm{np}} - \frac{\partial m/\partial \breve{q}}{W - (\partial L/\partial \breve{q})} = l_{\mathrm{np}} - \frac{m_{,\breve{q}}}{W - L_{,\breve{q}}} \tag{21}$$

By definition, the pitching moment about the neutral point does not vary with small changes in angle of attack, and from Eq. (4), we see that the change in the pitching moment about the center of gravity with respect to angle of attack is [3–7]

$$m_{,\alpha} = -l_{\mathrm{np}} L_{,\alpha} \tag{22}$$

Using Eq. (22) in Eq. (21) to express the location of the airplane's neutral point in terms of pitch stability, the stick-fixed maneuver point can also be located from

$$l_{\mathrm{mp}} = -\frac{m_{,\alpha}}{L_{,\alpha}} - \frac{m_{,\breve{q}}}{W - L_{,\breve{q}}} \tag{23}$$

The derivative $L_{,\breve{q}}$ is typically very small compared with the aircraft weight, and as will be shown, little or no accuracy is lost by ignoring this term to produce a result equivalent to that used in MIL-STD-1797A [9]:

$$l_{\mathrm{mp}} = -\frac{m_{,\alpha}}{L_{,\alpha}} - \frac{m_{,\breve{q}}}{W} \tag{24}$$

This result is consistent with the traditional short-period approximation [3–7], which also neglects $L_{,q}$.

Note that, although the stick-fixed maneuver point was defined in terms of the elevator angle per g, its location is completely independent of elevator power. Even if the elevator were completely fixed and unmovable, the location of the stick-fixed maneuver point defined by Eq. (21), (23), or (24) would not change. As we shall see, the stick-fixed maneuver point has greater significance than is suggested by its relation to the elevator angle per g.

V. Dynamic Length Scale and Dynamic Margin

The short-period natural frequency can also be expressed in terms of the dynamic time scale, which is defined by Eq. (19). Using the definitions $L_{,\breve{q}} \equiv L_{,q}g/V_o$, $L_{,\breve{\dot{\alpha}}} \equiv L_{,\dot{\alpha}}g/V_o$, $m_{,\breve{q}} \equiv m_{,q}g/V_o$, and $M \equiv W/g$ in Eq. (11), this approximation for the short-period natural frequency can be algebraically rearranged to give

$$\omega_n = \sqrt{\frac{L_{,\alpha}}{I_{yy}}\left(l_{\mathrm{np}} - \frac{m_{,\breve{q}}}{W - L_{,\breve{q}}}\right)\left(\frac{1 - L_{,\breve{q}}/W}{1 + L_{,\breve{\dot{\alpha}}}/W}\right)} \tag{25}$$

or after applying Eq. (22),

$$\omega_n = \sqrt{\frac{L_{,\alpha}}{I_{yy}}\left(-\frac{m_{,\alpha}}{L_{,\alpha}} - \frac{m_{,\breve{q}}}{W - L_{,\breve{q}}}\right)\left(\frac{1 - L_{,\breve{q}}/W}{1 + L_{,\breve{\dot{\alpha}}}/W}\right)} \tag{26}$$

Comparing Eqs. (23) and (26), we see that the short-period natural frequency is directly proportional to the square root of the distance that the stick-fixed maneuver point lies aft of the center of gravity

$$\omega_n = \sqrt{\frac{L_{,\alpha} l_{\mathrm{mp}}}{I_{yy}}\left(\frac{1 - L_{,\breve{q}}/W}{1 + L_{,\breve{\dot{\alpha}}}/W}\right)} \tag{27}$$

This proportionality was pointed out on page 186 of MIL-STD-1797A [9]. However, the relation has not been significantly used or widely acknowledged in the open literature on aircraft dynamics and handling qualities.

Using the relation $g \equiv W/M$, Eq. (27) can be rearranged to yield

$$\omega_n^2 = \frac{L_{,\alpha} l_{\mathrm{mp}} g}{I_{yy} W/M}\left(\frac{1 - L_{,\breve{q}}/W}{1 + L_{,\breve{\dot{\alpha}}}/W}\right) = \left(\frac{L_{,\alpha}}{W}\right)\left(\frac{l_{\mathrm{mp}} g}{I_{yy}/M}\right)\left(\frac{1 - L_{,\breve{q}}/W}{1 + L_{,\breve{\dot{\alpha}}}/W}\right) \tag{28}$$

The denominator in the second term on the far right-hand side of Eq. (28) has the units of length squared and it is exactly the square of what is commonly called the pitch radius of gyration for the airplane:

$$\text{pitch radius of gyration} \equiv r_{yy} \equiv \sqrt{I_{yy}/M} \tag{29}$$

The pitch radius of gyration defined by Eq. (29) is an important dynamic length scale associated with aircraft pitching motion [9,22,23]. A method for estimating the roll, pitch, and yaw radii of gyration in the early phases of airplane design is presented by Roskam [22] and is summarized by Raymer [23]. Using the definition given in Eq. (29), the short-period approximation given by Eq. (28) can be further rearranged to yield

$$\omega_n^2 = \left(\frac{L_{,\alpha}}{W}\right)\left(\frac{l_{\mathrm{mp}}}{r_{yy}}\right)\left(\frac{g}{r_{yy}}\right)\left(\frac{1 - L_{,\breve{q}}/W}{1 + L_{,\breve{\dot{\alpha}}}/W}\right) \tag{30}$$

Each of the first three ratios on the right-hand side of Eq. (30) is an important parameter associated with aircraft pitch dynamics and/or the associated aircraft handling qualities.

The first ratio on the right-hand side of Eq. (30) is traditionally called the

$$\text{acceleration sensitivity} \equiv L_{,\alpha}/W \tag{31}$$

Because the lift divided by the weight is equal to the load factor, the acceleration sensitivity is equal to the change in load factor with angle of attack. The short-period mode comprises primarily high-frequency oscillations in angle of attack and vertical displacement, with little change in forward velocity. Oscillations in angle of attack and the resulting oscillations in lift produce the normal acceleration associated with short-period motion. Thus, it is not surprising that the acceleration sensitivity is an important parameter associated with aircraft pitch dynamics. For an in-depth treatment of this topic, the reader is referred to Bihrle [10], as well as the United States military specifications MIL-F-8785C [8] and MIL-STD-1797A [9], which are summarized by Hodgkinson [24].

The second ratio on the right-hand side of Eq. (30) is an important dimensionless parameter, which is not typically discussed in the literature on aircraft dynamics and handling qualities. Here, we shall refer to this dimensionless parameter as the

$$\text{dynamic margin} \equiv l_{\mathrm{mp}}/r_{yy} \tag{32}$$

because it is a dynamic analogy to what is commonly called the static margin. The dynamic margin is formally defined here to be the distance that the stick-fixed maneuver point lies aft of the center of gravity, expressed as a fraction of the airplane's pitch radius of gyration. Because the distance aft from the center of gravity to the stick-fixed maneuver point and the pitch radius of gyration are both important characteristic length scales associated with pitch dynamics, the reader should not be too surprised to learn that aircraft handling qualities scale with this dynamic margin, not with the traditional static margin.

The third ratio on the right-hand side of Eq. (30) is simply the acceleration of gravity divided by the pitch radius of gyration. This parameter has the dimensions of inverse time squared, which are the same as those for the square of the short-period natural frequency that appears on the left-hand side of Eq. (30). This gravitational term appears in Eq. (30) as an artifact of rearranging Eq. (27) to expose the traditional acceleration sensitivity. The reason for rearranging Eq. (27) in this way will become apparent in the following section. The reader should note that the product of the first and third terms on the right-hand side of Eq. (30) is completely independent of the acceleration of gravity. Gravity affects the short-period natural frequency only through its effect on the maneuver point, which is seen in Eqs. (23) and (24).

As will be shown, the last term on the right-hand side of Eq. (30) has little or no significant effect on the short-period natural frequency of conventional piloted airplanes. The derivatives $L_{,\breve{q}}$ and $L_{,\breve{\dot{\alpha}}}$ are typically negligible compared with the aircraft weight. Thus, little accuracy is lost by ignoring these terms to produce a result equivalent to the traditional short-period approximation [3–7]

$$\omega_n^2 \cong \left(\frac{L_{,\alpha}}{W}\right)\left(\frac{l_{\mathrm{mp}}}{r_{yy}}\right)\left(\frac{g}{r_{yy}}\right) \tag{33}$$

which was used in MIL-STD-1797A [9].

VI. Control Anticipation Parameter and Dynamic-Margin Constraints

Extensive research on aircraft handling qualities [8–10] has shown that the effects of the short-period mode scale with what is commonly referred to as the control anticipation parameter (CAP). The CAP was originally defined by Bihrle [10] as the ratio of the initial pitch acceleration to the dimensionless steady-state load factor. However, the CAP is currently defined using an approximation to Bihrle's original definition, that is, the ratio of the short-period undamped natural frequency squared to the acceleration sensitivity:

$$\mathrm{CAP} \equiv \omega_n^2/(L_{,\alpha}/W) \tag{34}$$

For additional background on the CAP, the reader is referred to Moorhouse and Woodcock [11].

The United States military specifications MIL-F-8785C [8] and MIL-STD-1797A [9] show that producing acceptable airplane handling qualities depends in part on maintaining the CAP above a certain minimum value, which depends on both the desired level of pilot satisfaction and the complexity of the required flight phase. We can write this minimum-CAP constraint in the form

$$\omega_n^2/(L_{,\alpha}/W) \geq \dot{q}_c \tag{35}$$

where here we are using $\dot{q}_c$ to denote the variable CAP constraint, which has the dimensions of inverse time squared and was introduced by Bihrle [10] as being related to pitch acceleration. Both MIL-F-8785C and MIL-STD-1797A specify the minimum-CAP constraint relative to defined handling-quality levels and flight-phase categories

$$\dot{q}_c \equiv \begin{cases} \left.\begin{array}{l} 0.28\ \mathrm{s}^{-2}, \text{level 1} \\ 0.15\ \mathrm{s}^{-2}, \text{level 2} \end{array}\right\} \text{category A} \\ \left.\begin{array}{l} 0.085\ \mathrm{s}^{-2}, \text{level 1} \\ 0.038\ \mathrm{s}^{-2}, \text{level 2} \end{array}\right\} \text{category B} \\ \left.\begin{array}{l} 0.15\ \mathrm{s}^{-2}, \text{level 1} \\ 0.096\ \mathrm{s}^{-2}, \text{level 2} \end{array}\right\} \text{category C} \end{cases} \tag{36}$$

For the reader who may not be familiar with the military CAP requirements from which Eq. (36) was obtained, the definitions for handling-quality levels and flight-phase categories that are used in the military specifications are also summarized by Hodgkinson [24]. The level-1, category-A limit should be applied to demanding flight tasks such as air-to-air combat, aerobatics, and close-formation flying, which require rapid maneuvering and precise control. The level-1, category-B limit is used for cruise, climb, and other flight phases that are normally accomplished with gradual maneuvers without precision tracking. The level-1, category-C limit applies to takeoff, landing, and other flight phases that require accurate flight-path control with gradual maneuvering. The level-2 limits are usually considered to be acceptable only in a failure state.

The variable constraint used on the right-hand side of Eq. (35) and defined in Eq. (36) is a complex function of pilot limitations and flight-phase requirements. Students who are studying aircraft handling qualities commonly ask about the physical interpretation of this variable CAP constraint. As pointed out by Moorhouse and Woodcock [11], the interactions of a human pilot with the controls of an airplane are very complex and this question cannot be resolved conclusively. Even the physical origin of the CAP itself is sometimes misinterpreted because the associated terminology can be somewhat misleading. It is easy to see from Eq. (34) that the CAP is defined to be a function of only the short-period natural frequency, the change in lift with angle of attack, and the weight. None of these variables depend on the pilot's sensitivity to acceleration or on the pilot's ability to anticipate or control the airplane. It is actually the constraint on the right-hand side of Eq. (35) that depends on these factors, not the CAP itself. Thus, the minimum-CAP constraint is most likely affected by the pilot's limited sensitivity to pitch acceleration, as well as the importance of pitch acceleration to the task at hand. Nevertheless, for our present purpose, it is sufficient to say that the minimum-CAP constraint denoted here as $\dot{q}_c$ is an experimentally determined constraint having the dimensions of angular acceleration, which accounts for the combined effects of pilot limitations and flight-phase requirements.

It is interesting to note from Eqs. (35) and (36), that aircraft handling qualities as defined in the U.S. military specifications are actually independent of the parameter defined as the acceleration sensitivity in Eq. (31). Although it is argued in the military specifications that handling qualities should depend on acceleration sensitivity, we see from Eq. (30) that it is the short-period natural frequency that depends on $L_{,\alpha}/W$, not the CAP. The inclusion of $L_{,\alpha}/W$ in the definition of the CAP given by Eq. (34) was actually needed to remove $L_{,\alpha}/W$ from the short-period requirements. Combining Eq. (34) with Eq. (30) or the simpler approximation given by Eq. (33) reveals that the CAP is directly proportional to the acceleration of gravity divided by the radius of gyration squared, and not inversely proportional to the acceleration sensitivity:

$$\mathrm{CAP} = \frac{l_{\mathrm{mp}} g}{r_{yy}^2}\left(\frac{1 - L_{,\breve{q}}/W}{1 + L_{,\breve{\dot{\alpha}}}/W}\right) \cong \frac{l_{\mathrm{mp}} g}{r_{yy}^2} \tag{37}$$

This does not lessen the importance of $L_{,\alpha}/W$ as a dimensionless parameter associated with pitch dynamics. Furthermore, this does not imply that airplane handling qualities do not depend on the pilot's sensitivity to acceleration. With regard to $L_{,\alpha}/W$, the terminology "acceleration sensitivity" refers to the sensitivity of the airplane's normal acceleration to changes in angle of attack, not to the pilot's sensitivity to acceleration. Clearly, it is the constraint on the right-hand side of Eq. (35) that depends on the pilot's sensitivity to acceleration, not $L_{,\alpha}/W$ or the CAP itself. Thus, it might be more enlightening to think of $\dot{q}_c$ as an experimental constraint resulting from the pilot's pitch-acceleration-sensitivity limits as well as the flight-phase requirements.

Regardless of how the constraining parameter $\dot{q}_c$ is interpreted, after using the short-period approximation given by Eq. (30), the CAP constraint given by Eq. (35) can be translated to the dynamic-margin constraint:

$$\frac{l_{\mathrm{mp}}}{r_{yy}} \geq \frac{r_{yy}\dot{q}_c}{g}\left(\frac{1 + L_{,\breve{\dot{\alpha}}}/W}{1 - L_{,\breve{q}}/W}\right) \tag{38}$$

Values for $\dot{q}_c$ corresponding to the minimum-CAP requirement used in recent military specifications [8,9,24] are given by Eq. (36). Here again, the derivatives $L_{,\breve{q}}$ and $L_{,\breve{\dot{\alpha}}}$ are typically negligible compared with the aircraft weight, and for all practical purposes, Eq. (38) can be replaced with the very simple constraint

$$\frac{l_{\mathrm{mp}}}{r_{yy}} \geq \frac{r_{yy}\dot{q}_c}{g} \tag{39}$$

which is consistent with the traditional short-period approximation [3–7] used in MIL-STD-1797A [9].

It should be emphasized that the values for $\dot{q}_c$ given by Eq. (36) are the minimum values required to provide level-1 or level-2 handling qualities. These values should not be used as design goals. Whereas Eq. (36) gives only the lower CAP limits used by the military, there are upper CAP limits as well. These same military specifications show that, to maintain level-1 handling qualities in all flight phases, the CAP must be kept below 3.6 s^{-2}, and to maintain level-2 handling qualities, the CAP must be kept below 10 s^{-2}. The very best handling qualities are achieved for CAP values in the midrange, significantly above the level-1, category-A lower limit of 0.28 s^{-2} and significantly below the level-1 upper limit of 3.6 s^{-2}. It should also be noted that Eq. (39) does not typically provide the aft center-of-gravity limit for general aviation airplanes with reversible mechanical controls, because Federal Aviation Regulations Part 23 requirements [25] specify stability limits in terms of the control force derivatives and these requirements are typically more restrictive than Eq. (39) for light airplanes.

Because dynamic margin has such a profound effect on aircraft pitch dynamics, it is worth a closer look at the important dimensionless parameters that affect this characteristic length scale ratio. From Eq. (24), we can write

$$\frac{l_{\mathrm{mp}}}{r_{yy}} = -\frac{m_{,\alpha}}{r_{yy}L_{,\alpha}} - \frac{m_{,\breve{q}}}{r_{yy}W} \tag{40}$$

Because lift and weight are traditionally nondimensionalized with respect to the product of dynamic pressure and wing planform area, Eq. (40) suggests that we might define the

$$\text{dynamic moment coefficient} \equiv C_{\breve{m}} \equiv \frac{m}{\frac{1}{2}\rho V_o^2 S r_{yy}} \tag{41}$$

With this definition, the results presented in Eqs. (39) and (40) can be combined to yield

$$\frac{l_{\mathrm{mp}}}{r_{yy}} = -\frac{C_{\breve{m},\alpha}}{C_{L,\alpha}} - \frac{C_{\breve{m},\breve{q}}}{C_W} \geq \frac{r_{yy}\dot{q}_c}{g} \tag{42}$$

The dynamically scaled derivatives used in Eq. (42) are easily related to the traditional aerodynamically scaled derivatives commonly obtained from wind-tunnel tests. This requires knowing only V_o, g, r_{yy}, and the arbitrary reference length used to report the wind-tunnel results. From the traditional definitions of the aerodynamic pitch-stability and pitch-damping derivatives and the definitions given by Eqs. (20) and (41), it is easily shown that

$$C_{\breve{m},\alpha} = (c_{\mathrm{ref}}/r_{yy})C_{m,\alpha} \tag{43}$$

$$C_{\breve{m},\breve{q}} = [c_{\mathrm{ref}}^2 g/(2r_{yy}V_o^2)]C_{m,\bar{q}} \tag{44}$$

With these relations, Eq. (42) provides a simple but important fundamental constraint on pitch stability that could be used in the early phases of airplane design. This should provide a significant improvement over any constraint that is based on static margin alone.

It should be noted from Eq. (44) that, like the weight coefficient, the dynamic derivative $C_{\breve{m},\breve{q}}$ is inversely proportional to airspeed squared. With this knowledge, Eq. (42) displays the well-known fact that, in the absence of Mach and Reynolds number effects, the location of the stick-fixed maneuver point is independent of airspeed.

The reader should also note that the aft center-of-gravity limit specified by Eq. (42) is significantly affected by gross weight through its dependence on weight coefficient and the pitch radius of gyration. Airplanes can have fuel/cargo fractions as large as 50% or more. Because the variable fraction of the gross weight (fuel, stores, and cargo) will usually be concentrated close to the dry center of gravity, the radius of gyration defined by Eq. (29) will typically increase as fuel, stores, and cargo are removed. Hence, the critical aft center-of-gravity limit determined from the dynamic-margin constraint specified by Eq. (42) will typically be that for the minimum landing weight, not that for the maximum takeoff weight.

The mean aerodynamic chord is widely used as a reference length in aircraft flight mechanics literature. It should be emphasized that we are not proposing to discontinue use of the mean aerodynamic chord. To understand the results presented here, the reader must understand that the topic under consideration is not the length and time scales associated with the static forces and moments acting on an aircraft in steady flight. Our topic is flight dynamics, and our concern here is with the length and time scales associated with aircraft acceleration.

The constraint given by Eq. (42) was developed analytically from well-established dynamic stability criteria. This relation does not depend on the mean aerodynamic chord. However, by applying Eqs. (43) and (44), we can recast Eq. (42) in terms of the mean aerodynamic chord or any other arbitrary reference length:

$$\frac{l_{\mathrm{mp}}}{c_{\mathrm{ref}}} = -\frac{C_{m,\alpha}}{C_{L,\alpha}} - \frac{c_{\mathrm{ref}}g}{2V_o^2}\frac{C_{m,\bar{q}}}{C_W} \geq \frac{r_{yy}^2\dot{q}_c}{c_{\mathrm{ref}}g} \tag{45}$$

Equations (42) and (45) are mathematically identical. However, using the form given by Eq. (45) violates principals of dimensional similitude [26,27], which are commonly used to minimize the number of dimensionless variables in any nondimensional formulation. Note that Eq. (45) includes two additional dimensionless ratios, $c_{\mathrm{ref}}g/(2V_o^2) = \bar{\tau}/\breve{\tau}$ and r_{yy}/c_{ref}.

VII. Example Computations

The significance of the dynamic-margin constraint proposed in Eq. (42) is best demonstrated by example. Table 1 shows results obtained for the World War II British Spitfire Mark XVIII, which is typical of the airplanes being developed when early military requirements were first being defined. The aerodynamic data in Table 1 are based on results presented by Phillips [28]. The three flight cases listed in Table 1 are all for steady-level flight at standard sea level, with flaps and gear retracted. Case I is based on the airplane's dry weight and cases II and III are for two different airspeeds at the typical takeoff weight with internal fuel tanks full. The aerodynamic data used for cases I and II were extrapolated from the case III data using the Prandtl–Glauert compressibility correction [29]. Because this is a fighter-class airplane, which is responsible for category-A, B, and C flight phases, the level-1, category-A limit from Eq. (36) is constraining. Thus, the minimum-dynamic-margin limits for all cases shown in Table 1 are computed for a pitch-acceleration-sensitivity limit of $\dot{q}_c = 0.28\ \mathrm{s}^{-2}$. The dynamic-margin values listed in the next to the last row of Table 1 were computed from $-C_{\breve{m},\alpha}/C_{L,\alpha} - C_{\breve{m},\breve{q}}/C_W$ using the dynamic derivatives for the flight phase as specified by Eq. (42). The values listed in the last row are the minimum allowable values computed from the far right-hand side of Eq. (42), $r_{yy}\dot{q}_c/g$.

Table 2 shows similar results obtained for the Boeing 747-100. Results shown in Table 2 are based on data presented by Heffley and Jewell [30], which are summarized by Etkin and Reid [31]. All three flight cases are for steady-level flight at constant altitude. Case I is for landing configuration with 30 deg flaps, landing gear down, and an

Table 1 Computation of the static and dynamic margins for the Spitfire Mark XVIII

	Case I	Case II	Case III
Flight phase	Cat A	Cat A	Cat A
$\dot{q}_c$, s^{-2}	0.280	0.280	0.280
Altitude, ft	0	0	0
ρ, slug/ft^3	0.0023769	0.0023769	0.0023769
V_o, ft/s	130	140	440
Mach	0.12	0.13	0.39
S, ft^2	244	244	244
c_{ref}, ft	6.625	6.625	6.625
W, lbf	6865	8375	8375
I_{yy}, slug · ft^2	9795	10,060	10,060
r_{yy}, ft	6.78	6.22	6.22
C_W	1.4008	1.4735	0.1492
$C_{L,\alpha}$	4.5191	4.5241	4.8840
$C_{m,\alpha}$	−0.3503	−0.3507	−0.3786
$C_{m,\breve{q}}$	−7.172	−7.180	−7.751
$C_{\breve{m},\alpha}$	−0.3424	−0.3736	−0.4033
$C_{\breve{m},\breve{q}}$	−0.04424	−0.04162	−0.00455
$l_{\mathrm{np}}/c_{\mathrm{ref}}$	0.078	0.078	0.078
$l_{\mathrm{mp}}/c_{\mathrm{ref}}$	0.110	0.104	0.106
l_{mp}/r_{yy}	0.107	0.111	0.113
$r_{yy}\dot{q}_c/g$	0.059	0.054	0.054

Table 2 Computation of the static and dynamic margins for the Boeing 747-100

	Case I	Case II	Case III
Flight phase	Cat C	Cat B	Cat B
$\dot{q}_c$, s^{-2}	0.150	0.085	0.085
Altitude, ft	0	20,000	40,000
ρ, slug/ft^3	0.0023769	0.0012673	0.0005873
V_o, ft/s	221	518	871
Mach	0.20	0.50	0.90
S, ft^2	5500	5500	5500
c_{ref}, ft	27.31	27.31	27.31
W, lbf	564,000	636,600	636,600
I_{yy}, slug · ft^2	32.3×10^6	33.1×10^6	33.1×10^6
r_{yy}, ft	42.94	40.92	40.92
C_W	1.7667	0.6808	0.5196
$C_{L,\alpha}$	5.8760	4.6852	5.6092
$C_{m,\alpha}$	−1.4325	−1.1519	−1.6329
$C_{m,\breve{q}}$	−21.184	−20.688	−25.289
$C_{\breve{m},\alpha}$	−0.9110	−0.7688	−1.0899
$C_{\breve{m},\breve{q}}$	−0.12128	−0.02263	−0.00978
$l_{\mathrm{np}}/c_{\mathrm{ref}}$	0.244	0.246	0.291
$l_{\mathrm{mp}}/c_{\mathrm{ref}}$	0.352	0.296	0.319
l_{mp}/r_{yy}	0.224	0.197	0.213
$r_{yy}\dot{q}_c/g$	0.200	0.108	0.108

Table 3 Effects of weight on the dynamic margin for the Boeing 747-100

	Case I	Case II	Case III
Flight phase	Cat C	Cat C	Cat C
$\dot{q}_c$, s^{-2}	0.150	0.150	0.150
Altitude, ft	0	0	0
ρ, slug/ft^3	0.0023769	0.0023769	0.0023769
V_o, ft/s	221	221	221
Mach	0.20	0.20	0.20
S, ft^2	5500	5500	5500
c_{ref}, ft	27.31	27.31	27.31
W, lbf	491,400	564,000	636,600
I_{yy}, slug · ft^2	31.5×10^6	32.3×10^6	33.1×10^6
r_{yy}, ft	45.43	42.94	40.92
C_W	1.5392	1.7667	1.9941
$C_{L,\alpha}$	5.8760	5.8760	5.8760
$C_{m,\alpha}$	−1.4325	−1.4325	−1.4325
$C_{m,\breve{q}}$	−21.184	−21.184	−21.184
$C_{\breve{m},\alpha}$	−0.8611	−0.9110	−0.9561
$C_{\breve{m},\breve{q}}$	−0.11464	−0.12128	−0.12729
$l_{\mathrm{np}}/c_{\mathrm{ref}}$	0.244	0.244	0.244
$l_{\mathrm{mp}}/c_{\mathrm{ref}}$	0.368	0.352	0.339
l_{mp}/r_{yy}	0.221	0.224	0.227
$r_{yy}\dot{q}_c/g$	0.212	0.200	0.191

airspeed 20% above stall. Cases II and III are for two different cruise conditions with flaps and gear retracted. This large transport is responsible for category-B and C flight phases but is not responsible for category-A flight phases. The level-1, category-C limit from Eq. (36) applies to landing and the level-1, category-B limit applies to cruise. Thus, the minimum-dynamic-margin limits shown in Table 2 are computed using a pitch-acceleration-sensitivity limit of $\dot{q}_c = 0.15$ s^{-2} for case I and $\dot{q}_c = 0.085$ s^{-2} for cases II and III.

Table 3 shows results obtained for the Boeing 747-100 in landing configuration at three different landing weights, with the airspeed held constant. Notice that the dynamic margin is very nearly independent of aircraft weight. As weight is increased, the stick-fixed maneuver point is moved forward as a result of the second term on the right-hand side of Eq. (24). Because $m_{,\breve{q}}$ is always negative, an increase in weight decreases l_{mp}. This increase in weight also decreases the radius of gyration. Hence, there is little change in the ratio l_{mp}/r_{yy}. However, the decrease in radius of gyration caused by an increase in aircraft mass also decreases the minimum-dynamic-margin constraint on the right-hand side of Eq. (42). Thus, the minimum weight case is constraining.

The two airplanes used to obtain the results listed in Tables 1–3 are very different in both size and function. In their respective classes, both airplanes were highly successful, providing good handling qualities with adequate but not excessive pitch stability. From Table 1, we see that the Spitfire exhibits a static margin of 7.8% and a minimum dynamic margin of 10.7%. This static margin exceeds the traditional minimum guideline of 5% by a factor of about 1.6. Similarly, the Spitfire's dynamic margin exceeds the minimum constraint of 5.9% obtained from Eq. (42) by a factor of about 1.8. In general, the traditional minimum-static-margin guideline of 5% will give results very similar to Eq. (42) for airplanes with size and proportions similar to those of the fighter airplanes of World War II. On the other hand, the 5%-minimum-static-margin guideline does not scale well as the size and proportions of an airplane are substantially changed. This is demonstrated in Table 2. Whereas the static margin of the Boeing 747 exceeds the 5%-minimum guideline by a factor of nearly 5, for the critical case of landing, the minimum-dynamic-margin constraint from Eq. (42) is exceeded only by a factor of 1.1.

The increase in pitch radius of gyration that takes place as fuel, stores, and cargo are removed from an airplane is clearly demonstrated for the Boeing 747 in Table 3. This relation between gross weight and pitch radius of gyration holds for almost any airplane. For example, Fig. 3 shows the pitch radius of gyration, nondimensionalized with respect to fuselage length, plotted as a

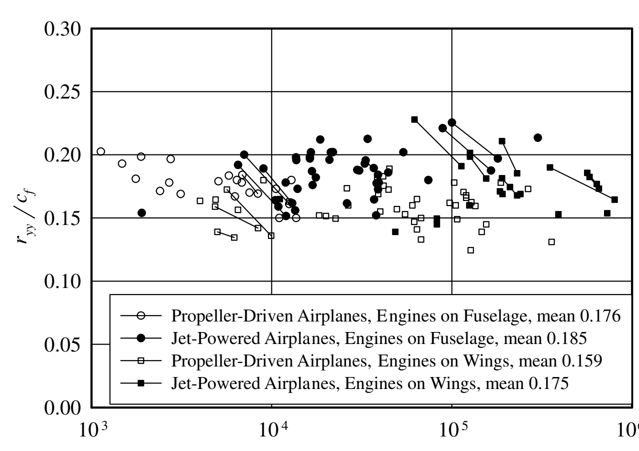


Fig. 3 Pitch radius of gyration for various airplanes nondimensionalized with fuselage chord length.

function of gross weight for a large number of widely different airplanes. The data used to generate Fig. 3 were obtained from several different sources [30–37]. For any airplane having data available at more than one gross weight, the corresponding points in Fig. 3 are connected with a solid line. In all such cases, the pitch radius of gyration is larger for the lightest loading.

The data plotted in Fig. 3 could be used to estimate the pitch radius of gyration for a particular type of airplane in the early phases of design. For example, the propeller-driven airplanes with fuselage-mounted engines, which are included in Fig. 3, have nondimensional pitch radii of gyration ranging from 0.150 to 0.202 with an arithmetic mean value of 0.176 and a standard deviation of 0.014. A lightly-loaded airplane typically has a pitch radius of gyration in the upper part of this range and data for heavily-loaded airplanes will usually fall in the lower part of this range.

If more is known about an airplane, its radius of gyration can be estimated more precisely from knowledge of the mass properties of similar airplanes. For example, the propeller-driven airplanes with fuselage-mounted engines that are included in Fig. 3 are composed of both fighters and general aviation airplanes. The nondimensional pitch radii of gyration for the fighters range from 0.150 to 0.184 with an arithmetic mean of 0.171 and a standard deviation of 0.012. On the other hand, the lighter general aviation airplanes exhibit nondimensional pitch radii of gyration ranging from 0.169 to 0.202 with a mean of 0.187 and a standard deviation of 0.013. At the other end of the spectrum, the jet transports with wing-mounted engines at maximum takeoff weight have nondimensional pitch radii of gyration ranging from 0.153 to 0.177 with a mean value of 0.164. For the transports of this type operating at minimum landing weight, the nondimensional pitch radii of gyration range from 0.190 to 0.228 and have a mean value of 0.208. For additional details on all data that were used to generate Fig. 3, the reader is referred to the original sources of the data [30–37]. For the purpose of comparison, Figs. 4 and 5 show the traditional nondimensional roll and yaw radii of gyration for the same airplanes that were used for Fig. 3. For the details of estimating airplane radii of gyration from such data, see Roskam [22], which includes three worked examples.

The traditional method that has been used to estimate the yaw radius of gyration in the early phases of airplane design is based on correlating data similar to that shown in Fig. 5, which is nondimensionalized with respect to the arithmetic average of the wingspan and the fuselage chord length. An alternate approach is to relate the yaw radius of gyration to the roll and pitch radii of gyration. By definition, we can write the relations

$$r_{xx}^2 \equiv \frac{1}{M}\iiint_M (y^2 + z^2)\,\mathrm{d}M \tag{46}$$

$$r_{yy}^2 \equiv \frac{1}{M}\iiint_M (x^2 + z^2)\,\mathrm{d}M \tag{47}$$

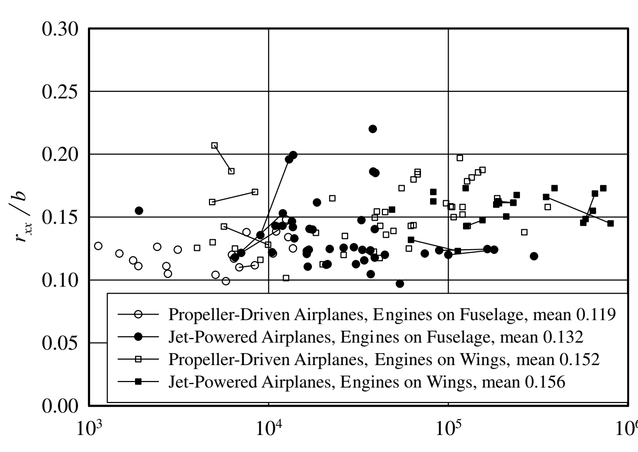


Fig. 4 Roll radius of gyration for various airplanes nondimensionalized with respect to wingspan.

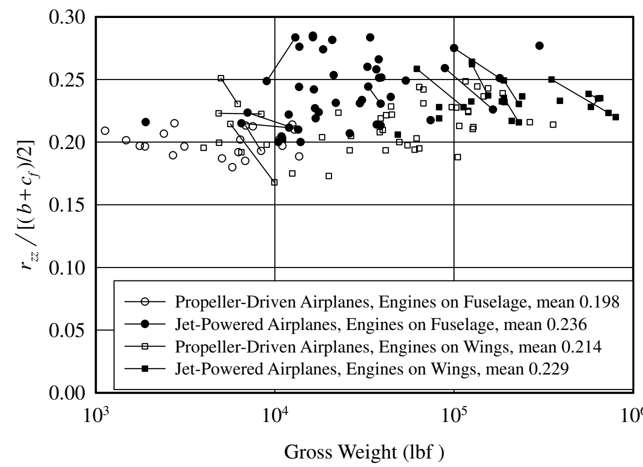


Fig. 5 Yaw radius of gyration for various airplanes nondimensionalized with respect to the average of the wingspan and the fuselage chord length.

$$r_{zz}^2 \equiv \frac{1}{M}\iiint_M (x^2 + y^2)\,\mathrm{d}M \tag{48}$$

From these definitions, it is easily shown that

$$r_{zz}^2 \equiv r_{xx}^2 + r_{yy}^2 - \frac{2}{M}\iiint_M z^2\,\mathrm{d}M \tag{49}$$

Because most of an airplane's mass is typically concentrated close to the x–y plane, Eq. (49) demonstrates that the yaw radius of gyration for an airplane is typically slightly less than the root square sum of the roll and pitch radii of gyration. This suggests that mass property data for the yaw radius of gyration may correlate better if we use the nondimensional parameter $r_{zz}/(r_{xx}^2 + r_{yy}^2)^{1/2}$. This is demonstrated in Fig. 6, which includes exactly the same data that are plotted in Fig. 5.

The results of the example computations that are presented in Tables 1–3 were obtained using the short-period approximation given by Eq. (33) with the maneuver margin obtained from the approximation given by Eq. (24). The development of this closed-form solution was based on three simplifying assumptions: 1) changes in forward velocity were neglected, 2) the contributions of drag to the normal force and pitching moment were neglected, and 3) changes in lift with respect to the angular rates q and $\dot{\alpha}$ were neglected. The accuracy of these assumptions can be quantitatively assessed by comparing results obtained from this approximation with more exact numerical solutions obtained from the full linearized

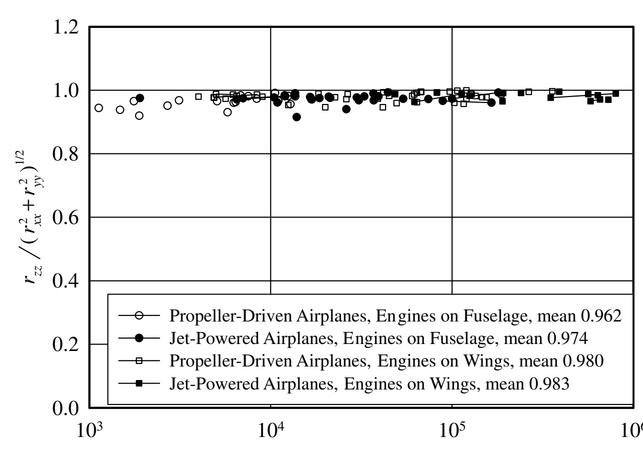


Fig. 6 Yaw radius of gyration for various airplanes nondimensionalized with respect to the root square sum of the roll and pitch radii of gyration.

longitudinal equations of motion. Errors associated with such comparisons are presented in Fig. 7a for 70 different cases encompassing widely different airplanes and operating conditions. The data used to generate Fig. 7 were obtained from several different sources [28,30–35]. For the 70 cases that are shown in Fig. 7a, the maximum positive error for Eqs. (24) and (33) relative to the full 3-degree-of-freedom numerical solution is $+1.90\%$, the maximum negative error is -1.38%, the mean error is $+0.31\%$, and the root-mean-square (rms) error is 0.69%.

For these 70 cases, Fig. 7b presents the errors associated with the traditional short-period approximation [3–7]

$$\omega_n = \sqrt{\frac{m_{,q}}{I_{yy}}\frac{Z_{,\alpha}}{MV_o} - \frac{m_{,\alpha}}{I_{yy}}} \tag{50}$$

which includes the contributions from drag but is otherwise identical to the approximation that is given by Eqs. (24) and (33). For these cases, the maximum positive error for the traditional short-period approximation is $+2.13\%$, the maximum negative error is -0.69%, the mean error is $+0.52\%$, and the rms error is 0.81%. For these conventional airplanes and operating conditions, the results presented in Fig. 7 clearly show that including the contributions of drag to the normal force and pitching moment does not improve the accuracy of the short-period approximation.

The effects of including the changes in lift with respect to the angular rates q and $\dot{\alpha}$ are presented in Figs. 7c and 7d. The errors shown in Fig. 7c are based on the short-period approximation given by Eq. (30) with the maneuver margin obtained from Eq. (23). For the same 70 cases used to obtain Figs. 7a and 7b, the maximum positive error shown in Fig. 7c is $+0.26\%$, the maximum negative error is -2.34%, the mean error is -0.35%, and the rms error is 0.65%. Similar results are shown in Fig. 7d for a short-period approximation that includes the effects of drag as well as the changes in lift with respect to the angular rates:

$$\omega_n = \sqrt{\left[\frac{m_{,q}}{I_{yy}}\frac{Z_{,\alpha}}{MV_o} - \frac{m_{,\alpha}}{I_{yy}}\left(1 + \frac{Z_{,q}}{MV_o}\right)\right] \Big/ \left(1 - \frac{Z_{,\dot{\alpha}}}{MV_o}\right)} \tag{51}$$

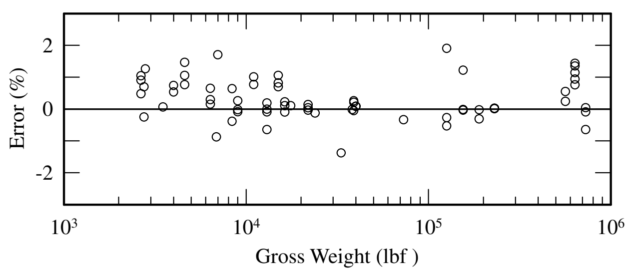
For this approximation, the maximum positive error is $+0.49\%$, the maximum negative error is -1.66%, the mean error is -0.13%, and the rms error is 0.42%. As demonstrated in Fig. 7, the accuracy of the short-period approximation is not improved by including the changes in lift with respect to the angular rates q and $\dot{\alpha}$. Results obtained from using Eq. (24) in Eq. (33) agree with more accurate numerical solutions to a precision that is within the uncertainty associated with the related aerodynamic derivatives.

VIII. Conclusions

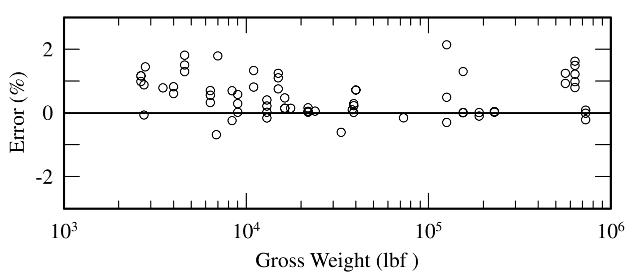
For many years, static-margin requirements have been used as design constraints in the attempt to provide adequate pitch stability for human-piloted airplanes. Although the 5%-static-margin requirement once used by the military has been replaced with dynamic requirements, static-margin constraints are still commonly used as preliminary guidelines for establishing good airplane handling qualities. The dynamic-margin constraint proposed in Eq. (42) provides a more realistic preliminary guideline for fixed-wing airplanes of all classes and size, which are being controlled by an onboard human pilot. It does not apply to unmanned aerial vehicles or any other aircraft under autopilot control, and it does not apply to any remotely piloted aircraft. It has been shown that setting $\dot{q}_c$ according to Eq. (36) and using the result in Eq. (42) provides a constraint that is equivalent to an important minimum-CAP requirement used in recent U.S. military specifications. Equation (42) also shows that the critical aft center-of-gravity limit specified by this dynamic stability constraint typically occurs at minimum gross weight.

It should be noted that the dynamic time scale defined by Eq. (19) and the pitch radius of gyration defined by Eq. (29) are the only characteristic time and length scales that appear in any of the dimensionless parameters used in Eq. (42). The wing chord length does not appear in the definitions for the dynamic margin, the dynamic moment coefficient, or the dynamic pitch rate. This does not mean that wing chord length does not affect the pitch dynamics of an airplane. The effects of wing chord length are included in Eq. (42) through its effects on $C_{L,\alpha}$, $C_{\breve{m},\alpha}$, and $C_{\breve{m},\breve{q}}$. For example, $C_{L,\alpha}$ decreases as the chord length is increased with the wing area held constant. Thus, the effects of wing chord length are included in Eq. (42) through its effects on the aerodynamic derivatives. Beyond this, wing chord length does not appear naturally in the aircraft longitudinal equations of motion and it should not be used to nondimensionalize these equations or any solutions produced there from, if we expect the aircraft handling qualities to scale with the resulting dimensionless parameters.

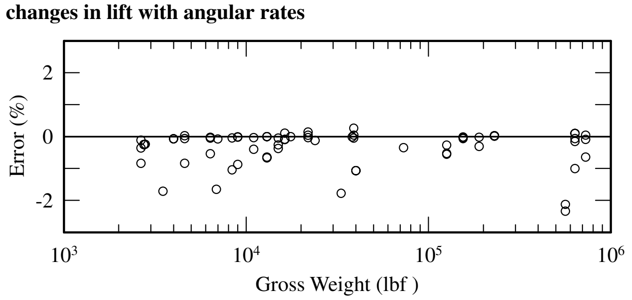
The mean aerodynamic chord has been widely used as the reference length for defining the traditional static margin and nondimensional pitch rate. However, thousands of hours of flight testing have shown that longitudinal aircraft handling qualities scale



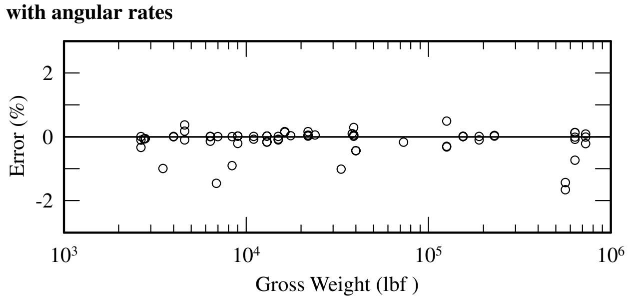
a) Equations (24) and (33), neglects drag and changes in lift with angular rates



b) Traditional approximations, Eq. (50), includes drag but neglects changes in lift with angular rates



c) Equations (23) and (30), neglects drag but includes changes in lift with angular rates



d) Generalized approximations, Eq. (51), includes drag and changes in lift with angular rates

Fig. 7 Natural-frequency errors associated with four different short-period approximations.

with the control anticipation parameter, not with the traditional static margin. Furthermore, as the analyses presented here show, the control anticipation parameter depends on pitch stability, pitch damping, and the pitch radius of gyration. It does not depend on the mean aerodynamic chord. Moreover, the inertial effects of a given pitch rate increase with airspeed. This dynamic characteristic is captured in the proposed dimensionless pitch rate defined by Eq. (20). This is in direct contrast with the traditional dimensionless pitch rate, which decreases with airspeed.

As seen in Eq. (21), the stick-fixed maneuver point for any airplane is aft of the stick-fixed neutral point by a distance that is proportional to the pitch damping. Any pitch-stability constraint that is based on static margin alone does not account for the fact that both pitch stability and pitch damping affect the longitudinal handling qualities of an airplane. When pitch damping is high, it is easier for a pilot to cope with less pitch stability. Using the dynamic-margin constraint that is proposed in Eq. (42) accounts for this effect. This relation predicts that, if the pitch damping were sufficiently large, a human pilot could fly an airplane having a zero or slightly negative static margin, even without the aid of a computer-controlled stability augmentation system.

In the early history of the development of airplane stability and control theory, Bryan [1] recognized that static stability was not an absolute requirement for human-piloted airplanes. As pointed out by Perkins [2], many of the early successful airplanes, including the 1903 Wright Flyer, had slightly negative static margins. Perkins suggested that the Wright brothers' success, in spite of a statically unstable airplane, was due to the fact that they "recognized that powerful controls would permit them to maintain the desired equilibrium." Equation (42) shows that their success was also supported in part by the ample pitch damping provided by the two large canards, which would have moved the stick-fixed maneuver point significantly aft of the stick-fixed neutral point. Their many hours of practice with the 1902 glider would have surely reduced the pilot's pitch-acceleration-sensitivity limit $\dot{q}_c$, which appears on the far right-hand side of Eq. (42). However, without pitch damping, no amount of practice would have made the 1903 Wright Flyer successful. Both history and mathematics show that a human pilot can fly a statically unstable airplane, if the pitch damping is sufficient to provide an adequate dynamic margin.

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